On the analog of crystallographic symmetry in the plane.

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Several authors have treated regular systems of points, regular divisions of the plane, and so on – in short, the analog of crystallographic symmetry in the plane. To my knowledge however two closely related topics of some interest have not yet been addressed:

1. The classification of the symmetries from the group-theoretical point of view, on which Schönflies's investigations and textbook are based.

2. The importance of these symmetries for art and art history; for the topic is really the symmetry of ornaments arranged periodically in the plane, such as textile and wallpaper patterns, parquetry, etc., which everyone is familiar with.

One can regard the symmetries of a "wallpaper pattern" arranged in the plane in two ways. One considers either:

1. motions of the plane in itself and reflections in mirror and glide-mirror planes perpendicular to it (an "opaque" plane),
2. motions of the plane in space (a "transparent" plane),

namely, such motions and reflections as bring the wallpaper pattern into alignment with itself. The first way is perhaps in many respects more natural; I use the second because of several conveniences.

The aligning motions of the wallpaper pattern can be one of four types: 1. Translation parallel to the plane. 2. Rotation around an axis perpendicular to the plane that cuts the plane in the center of rotation. 3. A flip (rotation of 180°) around a flip axis lying in the plane. 4. A screw motion with a rotational component of 180° around a glide axis lying in the plane.

The totality of the aligning motions of the wallpaper pattern form its group. All of the translations contained in the group are generated by two independent, therefore not parallel, translations: in this there is no difference between the groups. The geometric shape of the lattice generated by the translations can however vary: the fundamental region can be a general parallelogram, a rectangle, a rhombus, a square, or be made up of two equilateral triangles; by this classification criterion there are thus five types of patterns.

If we consider the rotations amongst the aligning motions, we find differences between the groups. The rotational centers can, as is well known, only be 2-, 3-, 4- or 6-fold, and they can also combine with flips; thus, by the implied classification criterion, we find ten classes (which correspond to the 32 crystal classes). I denote them with

$$C_1, \quad C_2, \quad C_3, \quad C_4, \quad C_6,$$
$$D_1, \quad D_2, \quad D_3, \quad D_4, \quad D_6.$$

C_1 means that only translations occur, D_1 means a single flip; for $n \geq 2$, C_n means the cyclic, D_n the dihedral group with an n-fold axis perpendicular to the plane.

Finally, one can choose the structure of the group as the classification criterion. Here only the types and relationships of the symmetry elements play a role; the metric specialization of the translational subgroup (to one of the five fundamental regions) only comes so far into view as is required by the other symmetry elements.

I found that only 17 groups with different structures exist in the plane.

Only one group belongs to each of the classes C_1, C_2, C_3, C_4, C_6.

3 groups belong to the class D_1:

- D_1ff: fundamental region rectangle, only flip axes;
- D_1gg: ″ ″ ″ glide axes;
- D_1fg: ″ rhombus, alternating flip and glide axes.

4 groups belong to the class D_2:

a) with rectangular fundamental region:

- D_2ffff: only flip axes parallel to both rectangle sides;
- D_2gggg: ″ glide axes ″ ″ ″ ;
- D_2ffgg: flip axes parallel to one rectangle side, glide axes to the other.

b) with rhombus as fundamental region:

- D_2fgfg: alternating flip & glide axes parallel to both rhombus diagonals.

2 groups belong to the class D_4:

- D_4^*: flip axes through all 4-fold rotational centers;
- D_4°: ″ ″ no ″ ″

2 groups belong to the class D_3:

- D_3^*: flip axes through all of the 3-fold rotational centers;
- D_3°: ″ ″ ⅓ ″ ″ ″

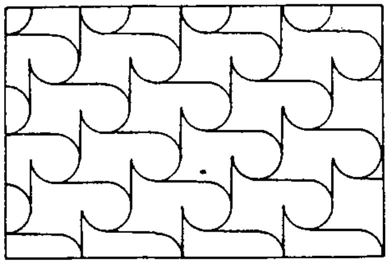
Only 1 group belongs to the class D_6.

In total: 5+3+4+2+2+1=17 groups in the plane.

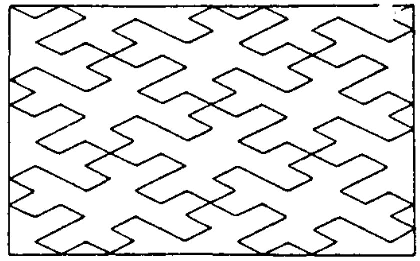
Let us compare the results with those in space:

	Plane	Space
Lattices	5	14
Classes	10	32
Groups	17	230

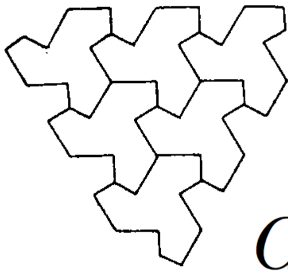
To carry out the proof of the completeness of the enumeration is, for someone familiar with Schönflies's investigations and textbook, only an exercise. I do not set out my proof here, as it does not appear sufficiently rounded to me.



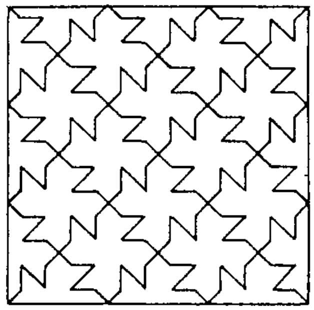
C_1



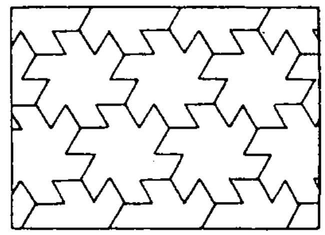
C_2



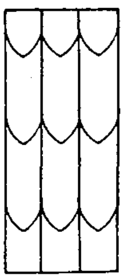
C_3



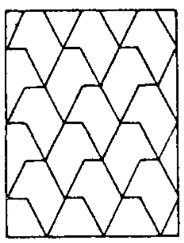
C_4



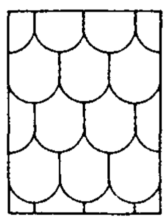
C_6



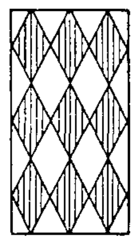
D_1ff



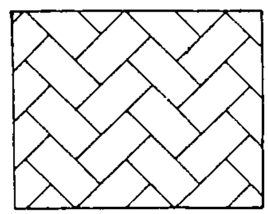
D_1gg



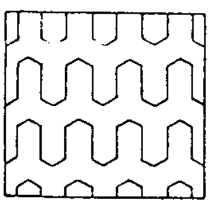
D_1fg



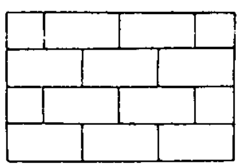
D_2ffff



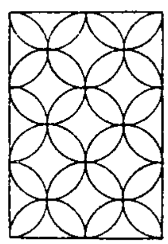
D_2gggg



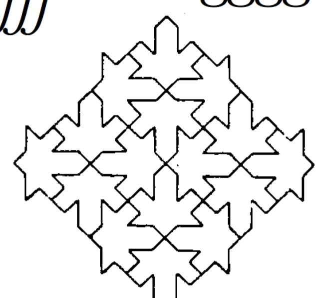
D_2ffgg



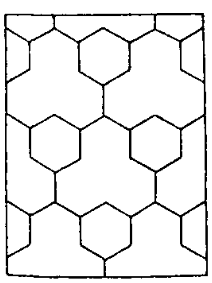
D_2fgfg



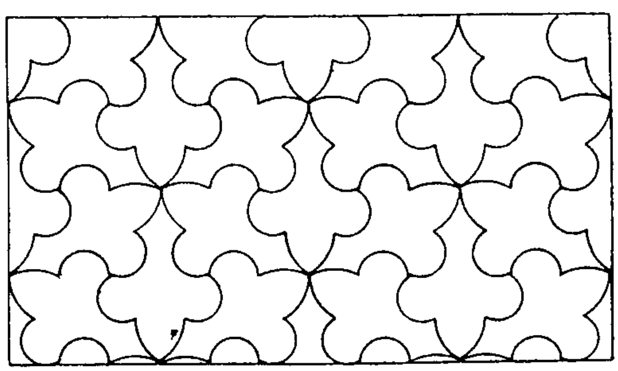
D_4^*



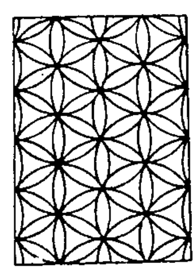
D_4°



D_3^*



D_3°



D_6

I give 17 ornament patterns to exemplify the 17 groups. For 4 groups, namely for D_2ffff, D_4^*, D_3^*, D_6, the boundary of the fundamental region is uniquely determined, because it consists entirely of flip axes. The figures for the other 13 groups actually show the division of the plane into fundamental regions, but in order to obtain pleasing patterns a few border lines are omitted. The resulting figures are divisions of the plane into congruent parts; the individual parts are not always fundamental regions, but rather combinations of multiple fundamental regions, which, however, show especial symmetry. For example the single tile in the picture of C_n is made up of n fundamental regions, which lie around an n-fold rotational center in cyclic order, $n = 1, 2, 3, 4, 6$. In the picture of D_2ffgg, there are even infinitely many fundamental regions combined in one band; and so on. The figures for C_1, C_3, C_4, D_1gg are ad hoc inventions. In order to incorporate everyday examples, D_1fg, D_2fgfg, D_2gggg are illustrated through the most ordinary brick and parquetry patterns. The other figures are schemata of traditional ornaments of diverse art-historical origin.

Incidentally the study of ornaments also supplies other mathematical problems. A particularly easy exercise is the following: to enumerate all the different symmetry structures of a border (band, frieze); there are seven. That the mathematical study of ornaments also has some interest from the artistic point of view, I will discuss elsewhere.

Translated by Marius Kempe.

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Notes:

1. Unbeknownst to Pólya, the wallpaper groups had already been classified by Fedorov in 1891.
2. Throughout, I have replaced Pólya's abbreviation *k* (for 'Um*k*lappung') with *f* ('flip').
3. The last sentence refers to a book Pólya planned but never finished, *The Symmetry of Ornament*.