Fejér's proof of a minimum property of the orthic triangle

Marius Kempe

> *[Fejér] could perceive the significance, the beauty, and the promise of a rather concrete, not too large problem … and, when he had found the solution, he kept on working at it with loving care, till each detail became fully intuitive and the connection of the details in a well-ordered whole fully transparent. It is due to such care spent on the elaboration of the solution that … most of his proofs appear very clear and simple.* George Pólya

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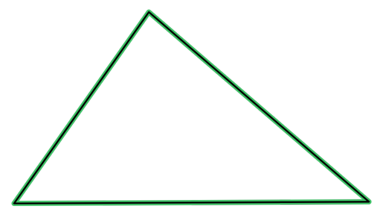
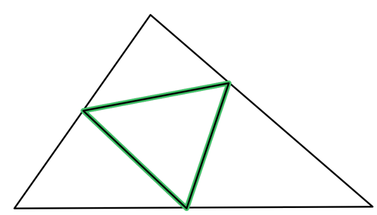
Of all the triangles inscribed in an acute-angled triangle, which has the shortest perimeter?

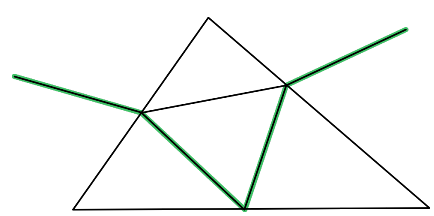
Let's start with an arbitrary acute-angled triangle,



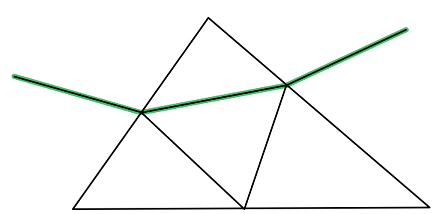
and inscribe some triangle into it.



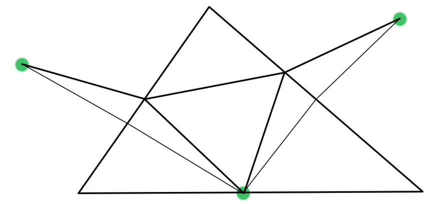
Now imagine that the left side of the outer triangle reflects the left side of the inner triangle like a mirror, and the same on the right.



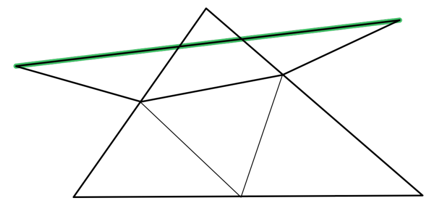
Obviously the reflections are just as long as the originals. Thus, this path is just as long as the perimeter of the inner triangle.



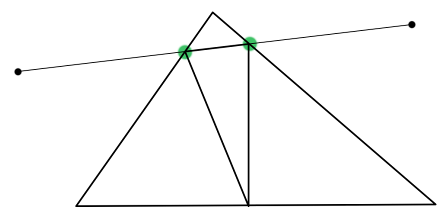
In fact, the locations of the ends of this path don't depend on where the top points of the inner triangle are – just where the bottom point is, because the mirrors mirror the point to the same places no matter where the sides go.



Now, the perimeter will be minimized when the path is shortest, which is obviously when it is a straight line between the two reflected points, since a straight line is the shortest way between any two points.

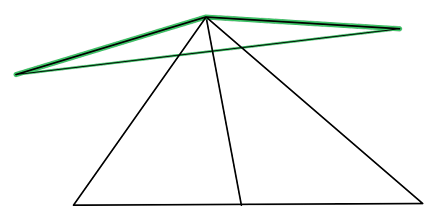


So, as long as the bottom point remains in one place, we know where to put the other two points – on the straight line in between the reflections of the bottom point in the left and right sides of the outer triangle.

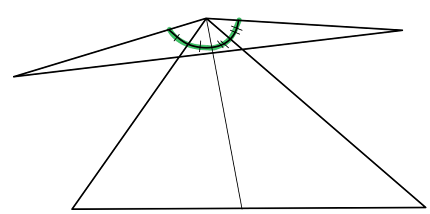


But where should the bottom point be?

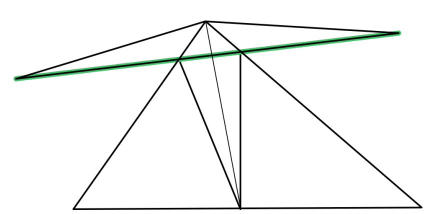
Note that if you pick any bottom point and reflect the line from it to the top of the outer triangle in the left and right sides of the outer triangle, the resulting triangle is isosceles (that is, it has two sides of the same length).



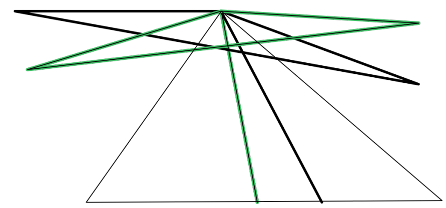
In this triangle the size of the top angle doesn't depend on where the bottom point is, because it's always double the top angle of the original outer triangle (since the mirrors always reflect the whole of the outer triangle's top angle, regardless of how it's broken in half).



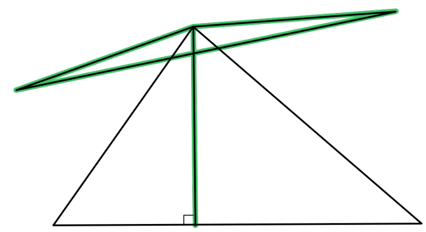
So now, since once we've picked the bottom point we know to put the other two points of the inner triangle onto the straight line between the reflections of the bottom point, we just want to place the bottom point so that the line is as short as possible – and that line is the base of the isosceles triangle.



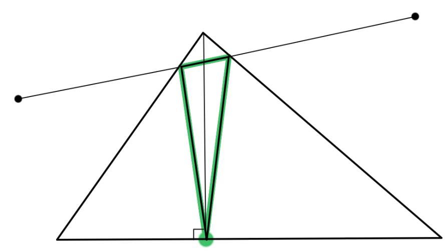
And since the top angle is always the same, we simply want to bring the base of the triangle as close to its top corner as possible, in other words, to make its sides as short as possible.



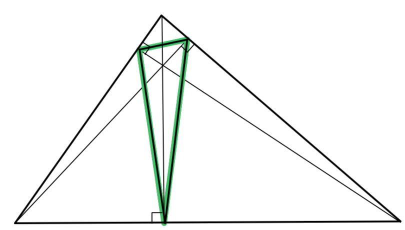
But of course its sides are as long as the line from the bottom point to the top of the outer triangle, because they are made by reflecting that line, and the shortest such line is the one that meets the bottom perpendicularly.



So the bottom point must be the point where the perpendicular line meets the bottom of the outer triangle.



Since this is the solution, and yet we could have used the same argument to reach it while starting with the left or right sides of the outer triangle as bottom, each corner of the inner triangle must be where the lines from each corner of the outer triangle meet its sides perpendicularly.



Such an inner triangle is called the *orthic triangle* of the outer triangle.

The epigraph is from Pólya's obituary of Fejér (*J. London Math. Soc.*, 1961, p. 501-506).
The visual approach was inspired by Casselman's review of Tufte's *Visual Explanations* (*Notices Amer. Math. Soc.*, January 1999).
The proof was first published in the second edition of Rademacher & Toeplitz's *Von Zahlen und Figuren* (Springer, 1933, ch. 6).
The problem was first posed and solved by G. F. Fagnano in 1775 (*Nova Acta Eruditorum*, p. 281-303).